

11/14/14.

Stable holomorphic bundles over Kähler surfaces. ①

Donaldson & Kronheimer Ch. 6

Today: review Donaldson's pt of N-S Thm and discuss $\dim_{\mathbb{C}} X = 2$ case. Fixed metrics.

Simplifying assumption: E hol' $SL(2, \mathbb{C})$ bundle, i.e. $\text{rk } E = 2$ and $1^2 E \cong \mathbb{C}$. hol. L -transformations are hol' mod $SL(2)$.

Unit-Vec: A - unitary conn. w/ curvature F_A of type $(1,1)$ ($\bar{\partial} A^2 = 0$).

defines integrable cpx str. $\bar{\partial} A = \bar{\partial} + A^{0,1}$ on E
 ($\Rightarrow$ any conn on $\text{over } \mathbb{R}$ -surf defines hol. str.)

Chern: Given hol. str. $\bar{\partial}$, get unique unitary

conn $\partial + A = \bar{\partial}_A + \bar{\partial}$. (s.t. $\mathbb{C}$ -linear.
 $\bar{\partial}_A^2 = 0$, $A_{\bar{\partial}}^{\alpha} \beta = H^{\alpha \bar{\beta}} \bar{\partial}_{\bar{\beta}} H^{\gamma \bar{\delta}} \bar{\partial}_{\bar{\delta}} \beta = H^{\alpha \bar{\delta}} \bar{\partial}_{\bar{\delta}} \beta$

$SL(2, \mathbb{C})!$

$\mathcal{G} = \{ \text{unitary sections of } E \otimes E^* \} \subset \mathcal{G}^c = \{ \text{invertible sections of } E \otimes E^* \}$

Two conns $A, B \in \mathcal{A}^{0,1}$ define isomorphic hol' str.'s iff $\exists g \in \mathcal{G}$

s.t. $g \circ \bar{\partial}_A = \bar{\partial}_B \circ g \Rightarrow \bar{\partial}_B = g \bar{\partial}_A g^{-1} = \bar{\partial}_A - (\bar{\partial}_A g) g^{-1} : \text{same } \mathcal{G}^c\text{-orbit. in } \mathcal{A}^{0,1}.$
 $\bar{\partial}_B = \bar{\partial}_A + (\bar{\partial}_A g g^{-1})^*$
Equivalently: $\bar{\partial}_{A \otimes B} g = 0$.

(but abv. act on stupid sense which would give unitary conns for diff metrics)

So: $A''/g^c \simeq \left\{ \begin{array}{l} \text{hol'ic structures on fixed} \\ \text{C}^\infty \text{ complex bundle } E \end{array} \right\} \bigg/ \sim$ (w/ unitary metric) / (same)

same all metrics are equivalent topologically, could think of g acting on metrics alternately.

Topological. $SL(2, \mathbb{C})$ -bundles classified by c_2 alone... $\deg = c_2 = \mu = \frac{c_1(E)}{2}$

Write E for hol'ic bundle defined by $\bar{\partial}_A$.

Say $E \Rightarrow$ stable if any nonzero hol $E \rightarrow u$ (line bundle) has $\deg u > 0$.

Write $\hat{F}_A = \Lambda F_A = F_A \cdot \omega \in E^* \otimes E^*$.
($= *F$ in 2-d case.)

Assume E is indecomposable, i.e. $\text{rank}(E) = 2$.

Theorem: E is stable $\Leftrightarrow \exists g \in g^c$ s.t. $\Lambda F_{g(A)} = 0$.

(Equivalently, two diff't metrics on same hol'ic bundle.)

also g is unique in g^c/g .

Hermitian Yang Mills (implicity)

Proof in 1-D case: direct minimization of $J(A) = \|F_A\|^2$.

Lemma D. (Whitbeck): $A_i \in \mathcal{A}$ seq. of L^2 , conv w/ $\|F_{A_i}\|^2$

bounded. $\exists m_i \in L^2_2(g)$ s.t. $m_i(A_i) \rightarrow B$

weakly in $L^2_1(A_E)$. (on all of X .)

(I'll return to this.) Take a minimizing sequence $A_i \in g^c(A)$ and let $m_i(A_i) \rightarrow B$.

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Write $\bar{F}$ for the hol'c bundle defined
by $\bar{\partial}_B$ (on same top'cl bundle E). ^{By construction,} $\bar{J}(F) \in \bar{J}(E)$.

Lemma 1 (MAIN LEMMA): There exists a
[really the revolutionary contribution]
nonzero holomorphic map $E \xrightarrow{\bar{\partial}_A} \bar{\partial}_B F$.

Proof: Consider $\bar{\partial}_{AB}: \Omega^0(E \otimes E^*) \rightarrow \Omega^{0,1}(E \otimes E^*)$

$$\bar{\partial}_{AB} s = 0 \iff s \in \text{Hom}(E, F).$$

Assume the claim fails; i.e. $\bar{\partial}_{AB} s = 0 \Rightarrow s = 0$.

$\bar{\partial}_{AB}$ is elliptic! Fredholm alternative: (in fact whole
proof can
be seen
as such!)

$$\|\cdot\| = \|\cdot\|_{L^2}: \|s\|_{L^2} \leq C \|\bar{\partial}_{AB} s\|_{L^2} \quad \forall s \in \Omega^0(E \otimes E^*).$$

$$\Rightarrow \text{Sobolev: } \|s\|_{L^4} \leq C_1 \|s\|_{L^2} \leq C_1 \|\bar{\partial}_{AB} s\|_{L^2}.$$

$L^2 \hookrightarrow L^4$ compact $\Rightarrow A_i \rightarrow B$ in L^4 strongly.

$$(\bar{\partial}_{AB} - \bar{\partial}_{AA_i})s = (B - A_i) \otimes s \rightarrow 0 \text{ in } L^4.$$

$$\Rightarrow \|(\bar{\partial}_{AB} - \bar{\partial}_{AA_i})s\| \leq C_2 \|A_i - B\|_{L^4} \|s\|_{L^4}$$

$$\|s\|_{L^4} \leq C_1 \|\bar{\partial}_{AB} s\|_{L^2} \leq C_1 (\|\bar{\partial}_{AA_i} s\|_{L^2} + C_2 \|A_i - B\|_{L^4} \|s\|_{L^4})$$

$$(1 - C_1 C_2 \|A_i - B\|_{L^4}) \|s\|_{L^4} \leq C_1 \|\bar{\partial}_{AA_i} s\|_{L^2}. \quad (s \neq 0, \text{ large } \Rightarrow \text{Hom}(E, E_{A_i}) = 0 \quad \times)$$

Now: two cases (in Donaldson's statement of Lemma 7.)

(a) $E \rightarrow F$ has full rank somewhere (doesn't even need to be non-zero.)

one bundle
 $\Lambda^n E \rightarrow \Lambda^n F$ non-zero

TRIVIAL ANYWAY!

b/c same top bundle.

$\deg E = \deg F \Rightarrow \deg(\Lambda^n E \otimes \Lambda^n F^*) = 0$ (BASE CASE!)
 $\Rightarrow$ any holomorphic section is non-vanishing $\Rightarrow$ isomorphism!
 $E \rightarrow F$ isomorphism $\Leftrightarrow B \in h^0(A)$.

(b) E has nowhere full rank:

$E \rightarrow U \subset F$ - one bundle (our case)

Sub sheaf of F .

~~Do after~~ $U = \text{image}$

~~Modify $U = \text{image}(-E \otimes P)$ to make principal bundle (decrease degree...)~~

$\Rightarrow \deg U \leq \deg F$

Get $0 \rightarrow S \rightarrow E \rightarrow U \rightarrow 0$ exact.

Non-split extension (E decomposable.)

Stability: $\deg U = d > 0$
 $= -\deg P$ (for us.)

NEED LEMMA 2 here.

Lemma 3: Can decrease energy. Why?

Curvature increases in quotients
 decreases in subs

$F(A)|_U \leq F(A_U)$
 $F(A)|_S \geq F(A_S)$
 $=$ iff split.

Top'ly, $E = u \oplus \bar{u}$.
 Can change frame s.t. hol' c str. on E looks like:

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$A = \begin{pmatrix} A_s & \beta \\ -\beta^* & A_m \end{pmatrix}$ where $F(A_u) = -d\omega$
 $F(A_s) = -d\omega$ (Hodge theory)

β^0 splits?

$0 \neq \beta \in \Omega^{0,1}(U^* \otimes S)$ extension class. *de from G-11! Yang-Mills for hyper-planes*

Choose harmonic representative: $d\beta + d\beta^* = 0$.
 (change again by g)

$\Rightarrow F(A) = dA + A \wedge A$

$= \begin{pmatrix} F(A_s) - \beta \wedge \beta^* & 0 \\ 0 & F(A_m) - \beta^* \wedge \beta \end{pmatrix}$

oh my God this is what you get from the desired defn of Hodge star.

$= \begin{pmatrix} -d\omega - \beta \wedge \beta^* & 0 \\ 0 & -d\omega - \beta^* \wedge \beta \end{pmatrix}$
 $-|\beta|^2 d\bar{z} \wedge dz = |\beta|^2 dz \wedge d\bar{z} > 0$

$\Delta \bar{\partial} = \Delta \partial$
 $\Rightarrow d\beta = 0$
 $\Rightarrow d\beta^* = 0$

need!

$= \begin{pmatrix} (-) + (+) & 0 \\ 0 & (+) + (-) \end{pmatrix}$

$\|F(A)\|^2 \leq \|F(u)\| + \|F(s)\| \approx 2d$
 $\downarrow$ strict b/c $\beta \neq 0$.

Contradiction to $J(B) \leq J(A)$.

So to make into subbundle.

Careful. Look at form Holder.

Why: Say $F = u \oplus u^{-1}$, then $\|F(B)\|^2 \geq 2d$.

$0 + \omega: u = m(-\Sigma x_i)$
 $d = \deg u \leq \deg m$

Lemma 2: $0 \rightarrow m \rightarrow F \rightarrow K \rightarrow 0 \Rightarrow \|F(B)\| \geq 2 \deg m$
 $(+)(-)$ bigger total! $\Rightarrow$ splits (converse!)

Concl: $E \rightarrow F$! B is cons. in $g^c(A)$. (6)
 STABLE & CLOSED!

Direct minimization preserves stable g -orbits.

Hodge theory: $d_A^2 \bar{\partial} A h = i * F$ solvable (E indecomp.)
 $\bar{\partial} A = \text{GAUGE DIRAC} \rightarrow \text{GUY OPPOSITE TO } \langle \alpha, \cdot \rangle$
 $\Rightarrow$ can continue minimizing until $*F = 0$. $\square_{N-S}$ (uniqueness)

Now: $\dim_R X = 2$ ($\dim_R X = 4$)

Direct minimization?

Uhlenbeck: On $B_R \subset X$, if $\|F_A\|_{B_R} < \epsilon_0$

then $\exists n$ s.t. $\|u_n(A)\|_{L^2(B_{R/2})} \leq C \|F_A\|$

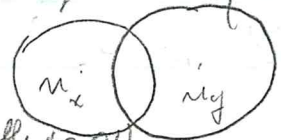
(Coulomb gauge: $d^* A = 0$, $dA + A \wedge A = F$)
 Continuity method. need to keep small! $\|A\|_{L^4} < 1$

Given A_i , $\|F_{A_i}\|_{L^2} < \epsilon$, $\exists i', \{x_i, \dots, x_m\} \subset X$

s.t. if $x \notin \{x_i\}$ then $\|F_{A_{i'}}\|_{L^2(B_{R_x}(x))} < \epsilon_0$, $i' > i_x$

$\Rightarrow$ Have weakly convergent seq. in $L^2(B_{R_x}(x))$.

But $L^2 \not\hookrightarrow C^0$! Oscill' of gauge transf's not controlled, cannot patch together without destroying bounds on A ! (either cannot cut off cleanly, or have to take balls too small)



MUST SAY THAT $\bar{\partial}$ and $\bar{\partial}^*$ ARE DEFINED!! BY THE FLOW OF SOLVE THE EQUATIONS. analysis can make algebra easier.

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Need higher regularity in order to even get a hol' comp

$\varepsilon \rightarrow \infty$ in the limit, and to deal with bubbling points etc.,

[Need A_ε to solve some PDE.]

TRY Yang-Mills gradient flow:

$$F_{A+a} = F_A + D_A a + a \wedge a$$

$$\Rightarrow \|F_{A+a}\|^2 = \|F_A\|^2 + 2(D_A a, F_A) + O(\|a\|^2)$$

$$\Rightarrow \text{Sty} = \boxed{D_A^* F_A = \partial_\varepsilon A.}$$

Energy inequality (defn): $\frac{d}{dt} \text{Ym} = - \int |D^* F|^2 dV \leq 0.$

$\Rightarrow$ Can take minimizing seq with $\|D_A^* F_{A_i}\| \rightarrow 0$
 & expect weak limit B with $D_B^* F_B = 0$.
 cfr. Kobler case, many other great properties.

Kobler identities: $\bar{\partial}_A^* = i[\partial_A, A]$

$$\partial_A^* = -i[\bar{\partial}_A, A]$$

Bianchi: $\partial_A F_A = \partial_A F_A + \partial_A F_A = 0$

$$\Rightarrow d_B^* F_B = i(\bar{\partial}_B - \partial_B) \hat{F}_B \Rightarrow \nabla_B \hat{F}_B = 0!$$

So if B exists, then it satisfies: (better than above! 2)

(a) $\hat{F}_B = 0 \Rightarrow F$ is stable on decomposable $F = u \otimes u^{-1}$, $\deg u = 0$.
 (Lemma 2)

(b) if $\hat{F}_B = (a, d)$, $F = u \otimes u^{-1}$, $\deg u = d > 0$.

$\chi(X) = 2\pi$

⑧

Still need to show:

- Lemma 0. Exists such an F [still mysterious], hol'c $SL(2, \mathbb{C})$ -bundle
 ~ Lemma 1. Exists a nonzero ^{global} hol'c map $E \rightarrow F$ (i.e. B is in some sense
 ~ (Lemma 2). $\textcircled{a} \Rightarrow E \rightarrow F$ is isom'ic (note one carry, $n \neq n-1$ clearly contradict.) in class of g'
 ~ Lemma 3. $\textcircled{b} \Rightarrow E \rightarrow F \rightarrow n^{-1}$ is nonzero, contradiction.

[Before we proceed] Note that [this all makes sense, because]

Derivative of g' -action: $\phi \rightarrow -\bar{\partial}\phi + \partial A \phi^*$

YMF: $\partial_t A = -\partial_A^* F_A = -i(\bar{\partial}_A - \partial A) F_A$

So moves along gauge direction $\phi = i\hat{F}_A!$

As long as flow exists, can write $\frac{\partial g}{\partial t} = i\hat{F}_A g_t$

$\Rightarrow A_t = g_t(A_0)$

In particular, flow preserves g^c -orbits. [as long as exists.]

Calculations:

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Weitzenböck on $\Omega_X^0(E)$:

$\partial = \partial_A$ etc.

Note $\bar{\partial}^* = i \wedge \partial$ on $\Omega^{0,1}$
 $\partial^* = -i \wedge \bar{\partial}$ on $\Omega^{1,0}$.

$$\Rightarrow \nabla^* \nabla = i \wedge (\partial - \bar{\partial})(\partial + \bar{\partial})$$

$$= i \wedge \{ \partial \bar{\partial} - \bar{\partial} \partial \}$$

$$\partial \bar{\partial} + \bar{\partial} \partial = F^{1,1}$$

$$i \wedge F^{1,1} = i \wedge \{ \partial \bar{\partial} + \bar{\partial} \partial \} = \hat{F}$$

$$\nabla^* \nabla = 2 \bar{\partial}^* \partial - i \hat{F}$$

$$= 2 \partial^* \bar{\partial} + 2 i \hat{F}$$

$$\partial_t \hat{F} = \wedge \partial_A (\partial_t A)$$

$$= i \wedge \partial_t (\bar{\partial} - \partial) \hat{F}$$

$$= i \wedge (\partial \bar{\partial} - \bar{\partial} \partial) \hat{F} = -\nabla^* \nabla \hat{F}$$

$$(\partial_t + \nabla^* \nabla) \hat{F} = 0$$

KEY TO

KAHLER CASE,

$$e = |\hat{F}|^2: \Delta e = 2 \langle \hat{F}, \nabla^* \nabla \hat{F} \rangle - |\nabla \hat{F}|^2 \Rightarrow (\partial_t - \Delta) e = -|\partial^* \hat{F}|^2.$$

$$\partial_t e = 2 \langle \hat{F}, \nabla^* \nabla \hat{F} \rangle$$

$$\Rightarrow \sup_X e_t \leq C \quad \forall t > 0.$$

Note: in dimension 4:

$$dz_1 \wedge dz_2 = (dx_1 + i dx_2) \wedge (dx_3 + i dx_4)$$
$$= dx_1 \wedge dx_3 - dx_2 \wedge dx_4 + i(dx_1 \wedge dx_4 + dx_2 \wedge dx_3)$$

self-dual.

$$d\bar{z}_1 \wedge d\bar{z}_2 = d\bar{z}_1 \wedge d\bar{z}_2, \text{ also}$$

$$dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 = (dx_1 + i dx_2) \wedge (dx_1 - i dx_2)$$
$$+ (dx_3 + i dx_4) \wedge (dx_3 - i dx_4)$$
$$= -2i(dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$$

self-dual

s-d: 3-d v. space:

So $A \in A^{1,1} \Rightarrow F^{0,2} = F^{2,0} = 0$

and $\hat{F} = 0 \Rightarrow F^+ = 0$: A is ASD instanton!

Along the flow on Kähler case: $|F^+| < C$.

My Thesis: "flow exists $\forall$ time!"

For any sequence $t_n \rightarrow \infty$, $\exists p, d=2$ and

$p_2(A(t_n)) \xrightarrow{\text{converge}} A_\infty$, Yang-Mills conn. on another hol. c. bundle F (removable sing's.)

$KEM \setminus \{x_1, \dots, x_n\}$



(11)

Limit preserves $SL(2, \mathbb{C})$ structure
and $A_\infty \in A^{1,1}$, as desired. $\square$

Have gauge tranfs $g_\alpha = g_{t_\alpha} \in \mathcal{G}^0$ ($g_\alpha(A(0)) = A(t)$).

$$\bar{\partial}_{AA_\alpha} g_\alpha = 0.$$

Assume $\det g_\alpha = 1$ and let $v_\alpha = \|g_\alpha\|$, $f_\alpha = \frac{1}{v_\alpha} g_\alpha$ ($\|f_\alpha\| = 1$).

Over $M \setminus \{x_i\}$, write $B_\alpha = f_\alpha(A_\alpha)$, $h_\alpha = f_\alpha \circ f_\alpha$ ($\|h_\alpha\| = 1$).

$$\Rightarrow \bar{\partial}_{A \circ B_\alpha} h_\alpha = 0 \quad \text{and} \quad B_\alpha \xrightarrow[\text{on } K]{O_\infty} A_\infty.$$

$$\Rightarrow h_\alpha \longrightarrow h: \mathcal{E}|_{X \setminus \{x_i\}} \longrightarrow \mathcal{F}|_{X \setminus \{x_i\}} \\ \text{w/ } \bar{\partial}_{AA_\infty} h = 0.$$

Hartogs: extends! (To show nonzero,

need AP est. on h_α , equiv'tly on $f_\alpha: \mathcal{E} \rightarrow \mathcal{E}$ (defined over X).

$$\text{Write } f = f_\alpha, \quad \bar{\partial}_{AA_\alpha} f = 0 \Rightarrow \bar{\partial}_{AA_\alpha}^* \bar{\partial}_{AA_\alpha} f = 0.$$

$$\text{Write } \tau = \text{Trace}(f^* f) \Rightarrow \|\tau\| = \|f\| = 1.$$

(just use f_α b/c h_α not defined away from K !)

$$\Delta \tau = T_2(\nabla^* \nabla(f^* f)) \quad , \quad \nabla = \nabla_A A_\alpha \quad (12)$$

$$= T_2((\nabla^* \nabla f^*) f + f^* \nabla^* \nabla f - 2 \nabla f^* \cdot \nabla f)$$

$$\leq T_2(\nabla^* \nabla f^* f + f^* (\nabla^* \nabla f))$$

Write: $\nabla^* \nabla f = \cancel{\partial^* \partial} f + i \hat{F} \# f$

Get: $\Delta \tau \leq C(\sup |\hat{F}_A| + \sup |\hat{F}_{A_\alpha}|) \cdot \tau$

$$\Delta \tau \leq C \cdot \tau$$

Lemma: $\int_x \tau = 1 \Rightarrow \int_{B_r} \tau \leq C r^3$

Proof: $(\int_x \tau^4)^{1/2} \leq \int_x |\nabla \tau|^2 \leq C \int_x \tau^2 = \tau^{2/3} \tau^{4/3}$

$$\leq C (\int \tau)^{2/3} (\int \tau^4)^{1/3}$$

$$\Rightarrow \int \tau^2 \leq C (\int \tau)^{2/3} \cdot (\int \tau^2)^{2/3}$$

$$\int_x \tau^4 \leq (\int \tau^2)^2 \leq C (\int \tau)^4 \leq C \int \tau \leq C$$

On B_R : $\int_{B_R} \tau \leq (\int \tau^4)^{1/4} (\int 1)^{3/4} \leq C r^3 \quad \square$

$\Rightarrow$ can choose $K = X \setminus B_r(x_1) \cup \dots \cup B_r(x_m)$ s.t.

$$\int_K |h_\alpha|^2 = \int_K |f_\alpha|^2 \geq \frac{1}{2} \Rightarrow h \neq 0!$$

(13)

Last step: (b) $h = h^+ \oplus h^- : \mathcal{E} \rightarrow u \oplus u, \deg u > 0$

and $i\hat{F}_{A_\infty} = \begin{pmatrix} -d & \\ & d \end{pmatrix} =: \Delta$. Need to show $h^- \neq 0$ in order to contradict stability.

Assume $\|g_t\| \rightarrow \infty$ else $\det h \neq 0$ (done.)

$\Rightarrow \exists t_\alpha \rightarrow \infty$ with $0 < \frac{d}{dt} \|g_t\|^2$

$$\frac{d}{dt} \|g_t\|^2 = 2 \int_X \text{Tr}(g_t^* i\hat{F} g_t)$$

as before $\int_{B_r(x)} \text{Tr}(g^* i\hat{F} g) \leq C r_\alpha^2 r^3 \quad (F|L|C)$

$$\leq \frac{1}{4} d r_\alpha^2 \quad (r \text{ suff. small})$$

$$\Rightarrow 0 < \left(\int_{X \setminus K} + \int_K \right) \text{Tr} g^* i\hat{F} g \leq \frac{1}{4} d r_\alpha^2 + \int_K \text{Tr} g^* i\hat{F} g$$

d suff. large, divide by $r_\alpha^2 \Rightarrow$

$$\frac{1}{4} d \leq \int_K \text{Tr} h_\alpha^* i\hat{F}_\alpha h_\alpha = \int_K h_\alpha^* \Delta \cdot h_\alpha + \varepsilon_\alpha \xrightarrow{0} 0$$

$$= d(\|h^-\|^2 - \|h^+\|^2) + \varepsilon_\alpha$$

$$\Rightarrow -\frac{1}{4} - \frac{\varepsilon_\alpha}{d} \leq \|h^-\|^2 - \|h^+\|^2$$

But $\|h\|^2 = \|h^+\|^2 + \|h^-\|^2 \geq \frac{1}{2} \Rightarrow +\frac{1}{4} - \frac{\varepsilon_\alpha}{d} \leq 2\|h^-\|^2$. $\square$ (b)

γ^c Orbit is unstable $\Leftrightarrow$ diverges under the flow.